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A linear approximation for the regular reflection of a weak shock at a wedge ☆

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Abstract

The problem of shock reflection by a wedge in the flow dominated by the unsteady potential flow equation is an important problem. In weak regular reflection, the flow behind the reflected shock is immediately supersonic and becomes subsonic further downstream. The reflected shock is transonic. Its position is a free boundary for the unsteady potential equation, which is degenerate at the sonic line in self-similar coordinates. Applying the special partial hodograph transformation used in [Zhouping Xin, Huicheng Yin, Transonic shock in a nozzle I, 2-D case, Comm. Pure Appl. Math. LVII (2004) 1-51; Zhouping Xin, Huicheng Yin, Transonic shock in a nozzle II, 3-D case, IMS, preprint, 2003], we derive a nonlinear degenerate elliptic equation with nonlinear boundary conditions in a piecewise smooth domain. When the angle between incident shock and wedge is small, we can see the weak regular reflection as the disturbance of normal reflection as in [Chen Shuxing, Linear approximation of shock reflection at a wedge with large angle, Comm. Partial Differential Equations 21(78) (1996) 1103–1118]. By linearizing the resulted nonlinear equation and boundary conditions with the above viewpoint in [Chen Shuxing, Linear approximation of shock reflection at a wedge with large angle, Comm. Partial Differential Equations 21(78) (1996) 1103–1118], we obtain a linear degenerate elliptic equation with mixed boundary conditions in a curved quadrilateral domain. By means of elliptic regularization techniques, a delicate a priori estimate and compact arguments, we show that the solution of the linearized problem is smooth in the interior and Lipschitz continuous up to the degenerate boundary.

Keywords: Degenerate elliptic equation; Shock wave; Potential equation; Regular reflection

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1. Introduction

If a plane shock hits a wedge, it will be reflected and a self-similar pattern of reflected shocks will appear when the shock moves forward in time. Such a problem is an important one in gas dynamics. It is also one of the basic models in studying the theory of weak solutions to the nonlinear mixed-type equations and the multidimensional Riemann problem for the quasilinear hyperbolic equations or systems in multidimensional spaces (see [3–5,27,28,32,33] and the references therein). There exists extensive literature on the study of a variety of patterns of reflected shocks by either numerical simulations or the analysis on the corresponding simplified equations (see [1,4,5,7,8,11–14,21] and so on), but there are few theoretical results on the unsteady full potential equation in the shock reflection. From the book [7] and Ref. [27], we know that the pattern of shock reflection depends on the angle of the wedge and the parameters of the incident shock. More precisely speaking, when the angle of the wedge is greater than a critical value determined by the incident shock, then the regular reflection occurs. Namely, if the incident shock with a constant speed hits the wedge with a large angle at the time $t = 0$, then for $t > 0$ the incident shock continuously travels forward with the same speed, meanwhile a reflected shock is formed. The reflected shock is immediately supersonic and becomes subsonic further downstream. Otherwise, if the angle of the wedge is less than the critical value, then the Mach reflection or several Mach configurations will happen (see [7,11,27] and so on). To determine the flow field behind the reflected shock, one needs to solve a nonlinear mixed-type equation. So far the rigorous mathematical theory has not been established. In order to treat this problem, some linearized methods of analysis on the simplified equation were developed. For example, when the wedge is of small angle and the incident shock is moderate-to-strong, Lighthill in [21] accounts that the reflected shock is approximately a solution of a linearized equation with a weak singularity on the sonic circle, moreover he uses the Busemann transformation to obtain a precise solution of the linearized problem. In the works [11–14], Blank, Harabetian, Hunter and Keller treated the reflected shock as so weak that it lies on a characteristic of the linearized equation. For the slightly stronger shocks, they use the weak nonlinear geometric optics theory to derive an asymptotic approximation. In [27], C.S. Morawetz also discussed approximation for the shock reflection problem in different scalings by taking the jump of the incident shock as a small parameter. If the angle of the wedge is near to π, S.X. Chen in [6] studied a related linear problem, with the position of the reflection shock fixed and the potential on the degenerate line given, meanwhile with the fixed boundary also replaced by a rigid wall. For this case, by use of the functional method in [29, Chapter I, Section 4], S.X. Chen established the existence of a H^1 weak solution (only continuous to degenerate boundary) to the linear problem as in [29, Theorem 1.4.1]. Obviously, the H^1 weak solution introduced in [29] is too weak; it is difficult to use it to treat the quasilinear equations.

In this paper, we discuss the regular reflection of a shock by a wedge $\{(x, y)\colon\ x \geqslant 0, -x/\operatorname{tg}\theta \leqslant y \leqslant x/\operatorname{tg}\theta\}$, here an angle $\theta > 0$ is very small. As in [11], we assume that the shock moves towards the wedge by a constant speed $\sigma > 0$ and reaches it at the time $t = 0$, moreover the gas ahead of the shock is at rest. Because the wedge is symmetric with respect to x-axis, it is enough to consider the upper half-plane $y > 0$ and a ramp $\{(x, y)\colon\ 0 \leqslant y \leqslant x/\operatorname{tg}\theta\}$ instead of the wedge. To study the regular shock reflection problem by a ramp, firstly we will apply the generalized partial hodograph transformation used in [30,31] to reformulate the corresponding nonlinear problem. Under the transformation, the position of reflected shock, the degenerate curve, two fixed boundaries $y = 0$ and $x = y\operatorname{tg}\theta$ are all known. Secondly, by use of the solution in the normal shock reflection as in [6], we linearize the resulted nonlinear equation and nonlinear boundary conditions in a domain Ω which is bounded by the sonic line L, $y = 0$, $x = y\,\mathrm{tg}\,\theta$ and the shock line. Then one obtains a linear degenerate elliptic problem in Ω. Note that the potential equation is hyperbolic above the sonic line L, so the corresponding solution (u_1, v_1, ρ_1) can be solved by an algebraic equation determined by the Rankine–Hugoniot condition and the entropy condition of the reflected shock. From [6], $(u_1, v_1, \rho_1) \approx (u_0, v_0, \rho_0)$, where (u_0, v_0, ρ_0) is the solution of normal reflected. Namely the curve L is actually known. To solve the linear degenerate elliptic equation and obtain the Lipschitz regularity of weak solution in the nonsmooth domain Ω, we will apply the elliptic regularization techniques and compact arguments. To achieve this, we have to give a very delicate a priori estimate on the weak solution in terms of the special coefficients in the equation and boundary conditions with classical elliptic theory (see [9,10,16,17] and the references therein). Here we should point out that our linear equation, boundary conditions and the regularity of boundary do not fit into the following forms discussed in [15,18,29]:

$$-\sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij}^2 u + \sum_{i=1}^{n} b_i(x)\partial_i u + c(x)u = f(x), \quad x \in Q,$$
$$u|_{\Sigma_2 \cup \Sigma_3} = g, \tag{1.1}$$

where $\partial Q \in C^2$, $a_{ij}(x), b_i(x), c(x), f(x) \in C^\alpha(\bar{Q})$ $(\alpha > 0)$, $g(x) \in C^1(\bar{Q})$ and $c(x) > 0$ is large enough. In addition, Σ_2 and Σ_3 represent the characteristic (degenerate) part and the noncharacteristic (degenerate or nondegenerate) part of ∂Q, respectively, where the Fichera function $b(x) < 0$ for $x \in \Sigma_2$. One should keep in mind that the largeness of c plays a key role in improving the smoothness of weak solution u. However, $c(x) \equiv 0$ in our problem, thus it seems difficult for us to use the approach in [15,18] or [29] to study the Lipschitz regularity of weak solutions for our problem.

Finally, we mention a notable work on the regular reflection of weak shocks in [4]. The authors S. Canic, B.L. Keyfitz and H.K. Eun prove the existence of a classical solution, which is continuous up to degenerate boundary, of the weak regular reflection problem near the degenerate curve for the unsteady transonic small disturbance (UTSD) model for shock reflection by a wedge. As indicated in [4,27], the UTSD model is only plausible near the degenerate line. In the general case, the reflected shock should be described by the unsteady full potential flow equation (when the Mach number does not exceed 1.3, it gives a good approximation; for details see [24] and many other references therein) or the complete compressible Euler system. However, as commented in [2], it seems difficult to find a transformation to lead the full potential equation for transonic flow to a tidy seconder-order equation for a velocity component as in UTSD, which is a quasi-linear equation with coefficients depending only on the unknown function u itself. Therefore, in order to study the global problem of the shock reflection, one has to treat the transonic full potential equation.

Our paper is organized as follows: In Section 2, firstly we give a mathematical description of the regular shock reflection problem. Next we reformulate the nonlinear equation and the nonlinear boundary conditions by use of the partial hodograph transformation as in [30,31]. In Section 3 we will give a detailed computation of the linearized problem. This yields the precise expressions for the coefficients, which are important in the subsequent a priori estimates. In Section 4, to overcome the difficulties caused by the degeneracy, we will use the technique of elliptic regularization. This derives a uniform elliptic equation depending on the small parameter $\varepsilon > 0$. Thanks to the special structure in the regularized equation and boundary conditions, we can obtain a priori estimates on the regularized solution and its first-order derivative, which is independent of the parameter ε. In Section 5 a compactness argument yields the proof on the main theorem.

2. Mathematical description and reformulation of problem

We assume that the time-dependent flow is, for $t > 0$, i.e., after the shock has hit the ramp, described by a self-similar potential flow; see for example [24] or [27]. From the potential condition that the velocity $(u, v) = \nabla\Phi = (\partial_x\Phi, \partial_y\Phi)$, we have only two equations, conservation of mass and Bernoulli's law. Namely, if ρ represents the density of the gas, then

$$\begin{aligned} &\rho_t + \operatorname{div}(\rho\nabla\Phi) = 0, \\ &\Phi_t + \frac{1}{2}|\nabla\Phi|^2 + h(\rho) = 0, \end{aligned} \tag{2.1}$$

where $h(\rho)$ is enthalpy and $h'(\rho) = c^2(\rho)/\rho$ with $c(\rho) = \sqrt{p'(\rho)}$ the speed of sound.

Suppose that the gas is polytropic and isentropic, namely the gas pressure p and density ρ have the relation $p = A\rho^\gamma$, here $A > 0$ and $1 < \gamma \leqslant 2$ are constants. In this case, one has

$$h(\rho) = \frac{\gamma p}{(\gamma-1)\rho}, \qquad c^2(\rho) = A\gamma\rho^{\gamma-1}. \tag{2.2}$$

Since $h'(\rho) > 0$, one then can define the inverse function of $h(\rho)$ as $H(s)$, namely,

$$\rho = H(D\Phi) = h^{-1}\left(-\Phi_t - \frac{1}{2}|\nabla\Phi|^2\right).$$

It follows from the conservation law of mass that

$$\big(H(D\Phi)\big)_t + \big(\Phi_x H(D\Phi)\big)_x + \big(\Phi_y H(D\Phi)\big)_y = 0. \tag{2.3}$$

Suppose that there is a uniform supersonic flow coming from infinity. Its parameters are $(u_\infty, 0, \rho_\infty)$ with $u_\infty > c_\infty$ and $c_\infty = c(\rho_\infty)$. As in [6,27], we assume that the shock moves towards the ramp $\{(x, y)\colon\ 0 \leqslant y \leqslant x/\operatorname{tg}\theta\}$ by a constant speed $\sigma > 0$ and reaches it at the time $t = 0$, moreover the gas ahead of the shock is at rest. If to denote the parameters of gas ahead of the shock $x = \sigma t$ by $(0, 0, \rho_+)$, then ρ_+ and σ will be determined by the Rankine–Hugoniot conditions and entropy condition as follows:

$$\sigma[\rho] - [\rho u] = 0, \qquad \sigma[\rho u] - \big[\rho u^2 + p(\rho)\big] = 0 \quad \text{and} \quad \rho_\infty > \rho_+. \tag{2.4}$$

By a simple computation one easily shows that (2.4) has a unique solution σ and ρ_+.

A self similar solution of (2.3) is of the form

$$\Phi = t\phi(\xi, \eta) \quad \text{with } \xi = \frac{x}{t} \text{ and } \eta = \frac{y}{t}. \tag{2.5}$$

In view of $H/H' = c^2(\rho)$ and (2.5), Eq. (2.3) can be reformulated as

$$A_{11}\phi_{\xi\xi} + 2A_{12}\phi_{\xi\eta} + A_{22}\phi_{\eta\eta} = 0, \tag{2.6}$$

where $A_{11} = c^2(\rho) - (\phi_\xi - \xi)^2$, $A_{12} = -(\phi_\xi - \xi)(\phi_\eta - \eta)$ and $A_{22} = c^2(\rho) - (\phi_\eta - \eta)^2$. It is easy to see that Eq. (2.6) is quasi-linear of mixed type with the type changing where $(\phi_\xi - \xi)^2 + (\phi_\eta - \eta)^2 = c^2(\rho)$.

If $(d\xi, d\eta)$ is tangent to the shock $\xi = s(\eta)$, then the continuous condition $[\phi] = 0$ and R–H condition $[H(\phi_\xi - \xi)] - [H(\phi_\eta - \eta)]s'(\eta) = 0$ on $\xi = s(\eta)$ may be rewritten as

$$\begin{cases} [\phi_\xi]\,d\xi + [\phi_\eta]\,d\eta = 0, \\ [H(\phi_\eta - \eta)]\,d\xi - [H(\phi_\xi - \xi)]\,d\eta = 0. \end{cases} \tag{2.7}$$

Finally, the velocity of the flow is tangent to the fixed boundaries $\eta = 0$ and $\xi = \eta \operatorname{tg}\theta$ of the ramp, so that

$$\begin{cases} \phi_\eta = 0 \quad \text{on } \eta = 0, \\ \phi_\xi - \phi_\eta \tan\theta = 0 \quad \text{on } \xi = \eta \tan\theta. \end{cases} \tag{2.8}$$

The function $\phi(\xi, \eta)$ satisfies the following condition:

$$\phi(\xi, \eta) = \phi_1(\xi, \eta) \quad \text{(be given) on } L, \tag{2.9}$$

here L is a smooth and known sonic curve which is completely determined by the parameters of the oblique shock reflection.

To study the nonlinear problem (2.6) with the boundary conditions (2.7)–(2.9), we will perform a partial hodograph transformation to fix the unknown shock introduced in [25,26] and so on. But for the convenience of computation, as in [30,31], we use the following partial hodograph transformation:

$$\begin{cases} x = -\dfrac{u_\infty(\eta \operatorname{tg}\theta - \xi)}{u_\infty(\eta \operatorname{tg}\theta - \xi) + \phi_\infty - \phi}, \\ y = \dfrac{\eta}{\eta_0}, \end{cases} \tag{2.10}$$

where $\phi_\infty(\xi, \eta) = u_\infty \xi - (h(\rho_\infty) + \frac{1}{2}u_\infty^2)$, $\eta_0 > 0$ is an appropriate large constant which will be chosen so that it gives us the convenience in the subsequent computation. As in [30,31], we will take $V = \eta \operatorname{tg}\theta - \xi + (\phi_\infty - \phi)/u_\infty$ as the new unknown function. Under the transformation (2.10), by the same computation as in [30], one can reformulate Eq. (2.6) as

$$a_{11}(X, V, \nabla V)\partial_{11}^2 V + 2a_{12}(X, V, \nabla V)\partial_{12}^2 V + a_{22}(X, V, \nabla V)\partial_{22}^2 V + F_0(X, V, \nabla V) = 0, \tag{2.11}$$

where $X = (x, y)$ and

$$\begin{aligned} a_{11}(X, V, \nabla V) &= A_{11}(\partial_\xi x)^2 + 2A_{12}\partial_\xi x \partial_\eta x + A_{22}(\partial_\eta x)^2 - \frac{A_{11}x(\partial_\xi x)^2\partial_1 V}{V + x\partial_1 V} \\ &\quad - \frac{2A_{12}x\partial_\xi x\partial_\eta x\partial_1 V}{V + x\partial_1 V} - \frac{A_{22}x(\partial_\eta x)^2\partial_1 V}{V + x\partial_1 V}, \\ a_{12}(X, V, \nabla V) &= \frac{1}{\eta_0}\left(A_{12}\partial_\xi x + A_{22}\partial_\eta x - \frac{A_{12}x\partial_\xi x\partial_1 V}{V + x\partial_1 V} - \frac{A_{22}x\partial_\eta x\partial_1 V}{V + x\partial_1 V}\right), \\ a_{22}(X, V, \nabla V) &= \frac{A_{22}V}{\eta_0^2(V + x\partial_1 V)}, \\ F_0(X, V, \nabla V) &= -\frac{2A_{11}(\partial_\xi x)^2(\partial_1 V)^2}{V + x\partial_1 V} - \frac{4A_{12}\partial_\xi x\partial_\eta x(\partial_1 V)^2}{V + x\partial_1 V} - \frac{2A_{22}(\partial_\eta x)^2(\partial_1 V)^2}{V + x\partial_1 V} \\ &\quad - \frac{2A_{12}\partial_\xi x\partial_1 V\partial_2 V}{\eta_0(V + x\partial_1 V)} - \frac{2A_{22}\partial_\eta x\partial_1 V\partial_2 V}{\eta_0(V + x\partial_1 V)}. \end{aligned}$$

Then the fixed boundary conditions (2.8) become into

$$-\frac{x\partial_2 V + \eta_0 \operatorname{tg}\theta}{V + x\partial_1 V}\partial_1 V + \partial_2 V = \eta_0 \operatorname{tg}\theta \quad \text{on } y = 0 \tag{2.12}$$

and

$$-\frac{\eta_0 + x\partial_2 V \operatorname{tg}\theta + \eta_0 \operatorname{tg}^2\theta}{V + x\partial_1 V}\partial_1 V + \operatorname{tg}\theta\,\partial_2 V = \eta_0 \operatorname{tg}^2\theta \quad \text{on } x = 0. \tag{2.13}$$

The shock condition (2.7) is actually equivalent to

$$G(X, V, \nabla V) = 0 \quad \text{on } x = -1, \tag{2.14}$$

where

$$\begin{aligned} G(X, V, \nabla V) = {} & \left\{\left(\frac{u_\infty(\eta_0 \operatorname{tg}\theta + x\partial_2 V)\partial_1 V}{\eta_0(V + x\partial_1 V)} - \frac{u_\infty \partial_2 V}{\eta_0} + u_\infty \operatorname{tg}\theta - \eta_0 y\right) H + \rho_\infty \eta_0 y\right\} \\ & \times \left\{\frac{(\eta_0 \operatorname{tg}\theta + x\partial_2 V)\partial_1 V}{\eta_0(V + x\partial_1 V)} - \frac{\partial_2 V}{\eta_0} + \operatorname{tg}\theta\right\} \\ & + \left\{\left(\frac{u_\infty \partial_1 V}{V + x\partial_1 V} + xV + \eta_0 y \operatorname{tg}\theta\right) H + \rho_\infty(u_\infty - xV - \eta_0 y \operatorname{tg}\theta)\right\} \\ & \times \left\{1 + \frac{\partial_1 V}{V + x\partial_1 V}\right\} \end{aligned}$$

with

$$\begin{aligned} H &= H\left(-\phi + (\xi\phi_\xi + \eta\phi_\eta) - \frac{1}{2}\big(\phi_\xi^2 + \phi_\eta^2\big)\right) \\ &= H\left(u_\infty V - \eta_0 u_\infty \operatorname{tg}\theta y + h(\rho_\infty) + \frac{u_\infty^2}{2}\right. \\ &\quad + \frac{u_\infty}{V + x\partial_1 V}\{\eta_0 y \operatorname{tg}\theta V + \eta_0 x y \operatorname{tg}\theta \partial_1 V - xV\partial_1 V - yV\partial_2 V\} \\ &\quad \left. - \frac{u_\infty^2}{2}\left(\frac{\partial_1 V}{V + x\partial_1 V}\right)^2 - \frac{u_\infty^2}{2\eta_0^2}\left(\eta_0 \operatorname{tg}\theta + \frac{\eta_0 \operatorname{tg}\theta \partial_1 V - V\partial_2 V}{V + x\partial_1 V}\right)^2\right). \end{aligned}$$

Finally, one believes that the potential ϕ is continuous across the degenerate line L, namely $\phi = \phi_1$ on L. Thus we conclude that V is also continuous on L. Denoting by

$$\tilde{V}_1(X) = \eta \operatorname{tg}\theta - \xi + \frac{\phi_\infty - \phi_1}{u_\infty} = \eta \operatorname{tg}\theta - \xi + \frac{\phi_\infty - \phi_0}{u_\infty} + O\big(\theta^2\big)$$

as in [6], we rewrite the degenerate boundary condition (2.9) in terms of V as follows:

$$V = \tilde{V}_1(X) \quad \text{on } L. \tag{2.15}$$

3. Linearization of the nonlinear problem (2.11)–(2.15)

We denote the parameters of the flow field behind the normal shock reflection by (u_0, v_0, ρ_0). Correspondingly, the potential and the sonic speed are written as $\phi_0(\xi, \eta)$ and $c_0 = c(\rho_0)$, respectively. As in Section 2, set

$$\begin{cases} V_0(X) = -\xi + \dfrac{\phi_\infty - \phi_0}{u_\infty}\dfrac{1}{u_\infty} = \left(h(\rho_0) - h(\rho_\infty) - \dfrac{u_\infty^2}{2}\right) > 0, \\ V_1(X) = V_0(X) + \eta_0 y \tan\theta. \end{cases} \tag{3.1}$$

Now, we focus on linearization of Eq. (2.11) and its boundary conditions (2.12)–(2.15) at $V_0(\theta = 0)$. As in [30,31], by a direct but very tedious computation we obtain the following linear equation and boundary conditions:

$$\begin{cases} a_{11}(x,y)\partial_{11}^2 \dot V + 2a_{12}(x,y)\partial_{12}^2 \dot V + a_{22}(x,y)\partial_{22}^2 \dot V = 0 \quad \text{in } \Omega, \\ \partial_2 \dot V = 0 \quad \text{on } y = 0, \\ \partial_1 \dot V = 0 \quad \text{on } x = 0, \\ \dot V = V_1 - V_0 \quad \text{on } L\colon\ y = l_0(x), \\ q\partial_1 \dot V + sy\partial_2 \dot V - s_0 \dot V = 0 \quad \text{on } x = -1, \end{cases} \tag{3.2}$$

where $\dot V = V - V_0$ and

$$a_{11}(x,y) = a^2 - x^2, \qquad a_{12}(x,y) = -xy, \qquad a_{22}(x,y) = b^2 - y^2,$$
$$s = \rho_0 - \rho_\infty + \frac{\rho_0 u_\infty V_0}{c_0^2}, \qquad q = \frac{\rho_0 u_\infty}{V_0}\left(1 - \frac{V_0^2}{c_0^2}\right) > 0,$$

with $a = c_0/V_0$, $b = c_0/\eta_0$.

One concludes that the degenerate line L can be approximately expressed as

$$y = l_0(x) = \sqrt{b^2 - \frac{b^2}{a^2}x^2}, \tag{3.3}$$

then

$$\Omega = \{(x,y)\colon\ -1 < x < 0,\ 0 < y < l_0(x)\}.$$

By [6] or Lemmas A.1, A.2, we have

$$s > 0, \qquad q = \frac{\rho_0 u_\infty}{V_0}\left(1 - \frac{V_0^2}{c_0^2}\right) > 0, \qquad V_0(\rho_0 - \rho_\infty) = \rho_\infty u_\infty, \qquad (a^2 - 1)s > q.$$

Subsequently, we will show that the weak solution exists for the problem (3.2), moreover it has the Lipschitz regularity up to the boundary.

For the simpleness, we write $\dot V$ in V in the other part of this paper.

4. The uniform estimates of solution to regularized problem of (3.2)

First, we consider following regular approximate problem:

$$\epsilon\left(x^2\partial_{xx}^2 V + 2xy\partial_{xy}^2 V + y^2\partial_{yy}^2 V\right) + \left(a^2 - x^2\right)\partial_{xx}^2 V - 2xy\partial_{xy}^2 V$$
$$+ \left(b^2 - y^2\right)\partial_{yy}^2 V - \epsilon V = 0 \quad \text{in } \Omega, \tag{4.1}$$
$$\partial_y V = 0 \quad \text{on } \bar\Sigma_1, \tag{4.2}$$
$$-\partial_x V + \epsilon\partial_y V - \epsilon V = 0 \quad \text{on } \bar\Sigma_2, \tag{4.3}$$
$$V = V_1 - V_0 \quad \text{on } \bar\Sigma_3, \tag{4.4}$$
$$q\partial_x V + sy\partial_y V - (s - q\epsilon)V = 0 \quad \text{on } \bar\Sigma_4, \tag{4.5}$$

where $\epsilon > 0$, $\Omega = \{(x,y)\colon\ -1 < x < 0,\ 0 < y < b\sqrt{1 - x^2/a^2}\}$ and

$$\Sigma_1 = \{(x, y)\colon\ -1 < x < 0,\ y = 0\}, \qquad \Sigma_3 = \left\{(x, y)\colon\ -1 < x < 0,\ y = b\sqrt{1 - \frac{x^2}{a^2}}\right\},$$

$$\Sigma_2 = \{(x, y)\colon\ x = 0,\ 0 < y < b\}, \qquad \Sigma_4 = \left\{(x, y)\colon\ x = -1,\ 0 < y < b\sqrt{1 - \frac{1}{a^2}}\right\}.$$

Then, for the fixed $\epsilon > 0$, (4.1) is uniformly elliptic equation in Ω, (4.2), (4.3), (4.5) are its discontinuous oblique boundary condition. From [19, Theorem 1] and [20, Theorem 4], problem (4.1)–(4.5) has a unique solution $V_\epsilon \in C^1(\bar{\Omega}) \cup C^2(\Omega)$.

Write $\Omega_\delta = \{(x, y) \in \Omega\colon\ \mathrm{dist}((x, y), \Sigma_3) \geqslant \delta > 0\}$. By the classical elliptical theories (see [9,10,16,17]), we have:

Lemma 1. *There is $M_1 > 0$, independent on ϵ, such that $|V_\epsilon| \leqslant M_1$.*

Lemma 2. *There is $M_2(\delta) > 0$, independent on ϵ, such that $|\nabla V_\epsilon| \leqslant M_2(\delta)$, if $(x, y) \in \Omega_\delta$.*

Lemma 3. *$V_\epsilon \in C^3(\overline{\Omega_\delta})$ and there is $M_3(\delta) > 0$, independent on ϵ, such that $|\nabla^3 V_\epsilon| \leqslant M_3(\delta)$, if $(x, y) \in \overline{\Omega_\delta}$).*

Lemma 4. *Let $v = |\nabla V_\epsilon|^2$, then the maximum value of v cannot be obtained in Ω.*

Proof. Computing $(4.1)_x V_{\epsilon x} + (4.1)_y V_{\epsilon y}$, we have

$$\sum_{i=1}^{9} I_i = 0 \quad \text{in } \Omega. \tag{4.6}$$

If $(x_0, y_0) \in \Omega$, and $v(x_0, y_0) = \max_{(x,y)\in\Omega} v > 0$, then

$$\begin{aligned}
I_1|_{(x_0,y_0)} &= \frac{1}{2}\epsilon\big(x^2 v_{xx} + 2xy v_{xy} + y^2 v_{yy}\big)\big|_{(x_0,y_0)} \leqslant 0,\\
I_2 &= \epsilon\big(-x^2 V_{\epsilon xx}^2 - 2xy V_{\epsilon xx} V_{xy} - y^2 V_{\epsilon xy}^2\big) = -\epsilon(x V_{\epsilon xx} + y V_{\epsilon xy})^2 \leqslant 0,\\
I_3 &= \epsilon\big(-x^2 V_{\epsilon xy}^2 - 2xy V_{\epsilon xy} V_{\epsilon yy} - y^2 V_{\epsilon yy}^2\big) = -\epsilon(x V_{\epsilon xy} + y V_{\epsilon yy})^2 \leqslant 0,\\
I_4|_{(x_0,y_0)} &= \frac{1}{2}\big(\big(a^2 - x^2\big)v_{xx} - 2xy v_{xy} + \big(b^2 - y^2\big)v_{yy}\big)\big|_{(x_0,y_0)} \leqslant 0,
\end{aligned}$$

by elliptic condition, one has

$$\begin{aligned}
I_5 &= \big(-\big(a^2 - x^2\big)V_{\epsilon xx}^2 + 2xy V_{\epsilon xx} V_{xy} - \big(b^2 - y^2\big)V_{\epsilon xy}^2\big) \leqslant 0,\\
I_6 &= \big(-\big(a^2 - x^2\big)V_{\epsilon xy}^2 + 2xy V_{\epsilon xy} V_{\epsilon yy} - \big(b^2 - y^2\big)V_{\epsilon yy}^2\big) \leqslant 0,\\
I_7|_{(x_0,y_0)} &= -\epsilon v|_{(x_0,y_0)} < 0.
\end{aligned}$$

Noticing $J_1 = (V_{\epsilon x} V_{\epsilon xx} + V_{\epsilon y} V_{\epsilon xy})|_{(x_0,y_0)} = 0$, $J_2 = (V_{\epsilon x} V_{\epsilon xy} + V_{\epsilon y} V_{\epsilon yy})|_{(x_0,y_0)} = 0$, we obtain that

$$\begin{aligned}
I_8|_{(x_0,y_0)} &= -2x(V_{\epsilon x} V_{\epsilon xx} + V_{\epsilon y} V_{\epsilon xy})|_{(x_0,y_0)} + 2\epsilon(V_{\epsilon x} V_{\epsilon xx} + V_{\epsilon y} V_{\epsilon xy})|_{(x_0,y_0)} = 0,\\
I_9|_{(x_0,y_0)} &= -2y(V_{\epsilon x} V_{\epsilon xy} + V_{\epsilon y} V_{\epsilon yy})|_{(x_0,y_0)} + 2\epsilon y(V_{\epsilon x} V_{\epsilon xy} + V_{\epsilon y} V_{\epsilon yy})|_{(x_0,y_0)} = 0.
\end{aligned}$$

Then $\sum_{i=1}^{9} I_i|_{(x_0,y_0)} < 0$, it is a contradiction to (4.6). Hence, we complete the proof of Lemma 4. □

Lemma 5. *Let $v = |\nabla V_\epsilon|^2$. If the positive maximum value of v can be obtained on $\Sigma_2 = \{(0, y)$: $0 < y < b\}$, then there exists $M_5 > 0$, independent on ϵ, such that $|v| < M_5$.*

Proof. If $(x_0, y_0) \in \Sigma_2$ and $v(x_0, y_0) = \max_{(x,y)\in\Sigma_2} v > 0$ then we have $v_y = 2(V_{\epsilon x}V_{\epsilon xy} + V_{\epsilon y}V_{\epsilon yy})|_{(x_0,y_0)} = 0$ and $v_x = 2(V_{\epsilon x}V_{\epsilon xx} + V_{\epsilon y}V_{\epsilon xy}) \geqslant 0$ if $(x, y) \in B_\delta((x_0, y_0)) \cap \Omega$ for some small $\delta > 0$. Letting $V_{\epsilon y}|_{(x_0,y_0)} \neq 0$, we obtain

$$\begin{cases} (V_{\epsilon x}V_{\epsilon xy} + V_{\epsilon y}V_{\epsilon yy})|_{(x_0,y_0)} = 0, \\ -V_{\epsilon xy} + \epsilon V_{\epsilon yy} - \epsilon V_{\epsilon y} = 0. \end{cases} \tag{4.7}$$

From (4.1), (4.3), and Lemma 3, we have

$$a^2 V_{\epsilon xx} + \left(b^2 - y^2 + \epsilon y^2\right)V_{\epsilon yy} - \epsilon V_\epsilon = 0 \quad \text{on } \Sigma_2. \tag{4.8}$$

Case 1. If $|V_{\epsilon y}|_{(x_\epsilon,y_\epsilon)} < M$, then the result is true by (4.3) and Lemma 1.

Case 2. If $\lim_{\epsilon\to 0}|V_{\epsilon y}|_{(x_\epsilon,y_\epsilon)} = +\infty$, by (4.3), we have

$$\lim_{\epsilon\to 0}\frac{V_{\epsilon x}}{V_{\epsilon y}} = 0. \tag{4.9}$$

Then

$$\begin{cases} V_{\epsilon xy} = -\dfrac{\epsilon V_{\epsilon y}^2}{\epsilon V_{\epsilon x} + V_{\epsilon y}}, \\ V_{\epsilon yy} = \dfrac{\epsilon V_{\epsilon x}V_{\epsilon y}}{\epsilon V_{\epsilon x} + V_{\epsilon y}}, \\ a^2 V_{\epsilon xx} = -\dfrac{\epsilon(b^2 - y^2 + \epsilon y^2)V_{\epsilon x}V_{\epsilon y}}{\epsilon V_{\epsilon x} + V_{\epsilon y}} + \epsilon V_\epsilon. \end{cases} \quad \text{on } (x_0, y_0) \in \Sigma_2. \tag{4.10}$$

From (4.8)–(4.10), for small $\epsilon > 0$, we have

$$\frac{a^2}{2\epsilon V_{\epsilon y}^2}v_x = \frac{a^2}{\epsilon V_{\epsilon y}^2}(V_{\epsilon x}V_{\epsilon xx} + V_{\epsilon y}V_{\epsilon xy}) < 0 \quad \text{on } (x_0, y_0).$$

This is impossible. We complete the proof of Lemma 5. □

Lemma 6. *Let $v = |\nabla V_\epsilon|^2$. If the positive maximum value of v can be obtained on $\Sigma_4 = \{(-1, y)$: $0 < y < b\sqrt{1 - 1/a^2} = d_0\}$, then there exists $M_6 > 0$, independent on ϵ, such that $|v| < M_6$.*

Proof. If $(x_\epsilon, y_\epsilon) \in \Sigma_4$ and $v(x_\epsilon, y_\epsilon) = \max_{(x,y)\in\Sigma_4} v > M_\epsilon \to \infty$ $(y_0 > d_0/2)$, then $v_x(x_\epsilon, y_\epsilon) \leqslant 0$. From the boundary condition, we have

$$V_{\epsilon x} = (my + n)V_{\epsilon y} + \mu V_\epsilon \quad \text{on } \Sigma_4, \tag{4.11}$$

where $m = -s/q < 0$, $n = 0$, $\mu = -m + \epsilon$. And

$$V_{\epsilon xy} = \epsilon V_{\epsilon y} + (my + n)V_{\epsilon yy} \quad \text{on } \Sigma_4. \tag{4.12}$$

By $v_y|_{(x_\epsilon,y_\epsilon)} = (2V_{\epsilon x}V_{\epsilon xy} + 2V_{\epsilon y}V_{\epsilon yy})|_{(x_\epsilon,y_\epsilon)} = 0$, we get

$$V_{\epsilon x}V_{\epsilon xy} + V_{\epsilon y}V_{\epsilon yy} = 0 \quad \text{on } (x_\epsilon, y_\epsilon). \tag{4.13}$$

From (4.11)–(4.13),

$$\begin{aligned} V_{\epsilon xy} &= \frac{\epsilon V_{\epsilon y}^2}{[(my+n)^2+1]V_{\epsilon y}+(-m+\epsilon)(my+n)V_\epsilon} \\ V_{\epsilon yy} &= -\frac{\epsilon V_{\epsilon x} V_{\epsilon y}}{[(my+n)^2+1]V_{\epsilon y}+(-m+\epsilon)(my+n)V_\epsilon} \end{aligned} \quad \text{on } (x_\epsilon, y_\epsilon). \tag{4.14}$$

By (4.1), on Σ_4, we have

$$\left(a^2-1+\epsilon\right)V_{\epsilon xx} = \epsilon V_\epsilon - 2(1-\epsilon)yV_{\epsilon xy} - \left(b^2-y^2+\epsilon y^2\right)V_{\epsilon yy}, \tag{4.15}$$

then, from (4.14), (4.15), on $(x_\epsilon, y_\epsilon) \in \Sigma_4$

$$\begin{aligned} \frac{1}{2}\left(a^2-1+\epsilon\right)v_x &= \left(a^2-1+\epsilon\right)V_{\epsilon x}V_{\epsilon xx} + \left(a^2-1+\epsilon\right)V_{\epsilon y}V_{\epsilon xy} \\ &= \epsilon V_\epsilon V_{\epsilon x} - \frac{2y(1-\epsilon)\epsilon V_{\epsilon y}^2 V_{\epsilon x}}{[(my+n)^2+1]V_{\epsilon y}+(-m+\epsilon)(my+n)V_\epsilon} \\ &\quad + \frac{\epsilon(b^2-y^2+\epsilon y^2)V_{\epsilon x}^2 V_{\epsilon y}}{[(my+n)^2+1]V_{\epsilon y}+(-m+\epsilon)(my+n)V_\epsilon} \\ &\quad + \frac{(a^2-1+\epsilon)\epsilon V_{\epsilon y}^3}{[(my+n)^2+1]V_{\epsilon y}+(-m+\epsilon)(my+n)V_\epsilon}. \end{aligned} \tag{4.16}$$

Case 1. If $|V_{\epsilon x}|_{(x_\epsilon,y_\epsilon)} < M$ or $|V_\epsilon|_{(x_\epsilon,y_\epsilon)} < M$, then the result is true by (4.11) and $y_\epsilon > d_0/2$.
Case 2. If $\lim_{\epsilon\to 0}|V_{\epsilon x}|_{(x_\epsilon,y_\epsilon)} = +\infty$, $\lim_{\epsilon\to 0}|V_{\epsilon y}|_{(x_\epsilon,y_\epsilon)} = +\infty$, by (4.11), we have

$$\lim_{\epsilon\to 0}\frac{V_{\epsilon x}}{V_{\epsilon y}} = my+n. \tag{4.17}$$

From (4.16), (4.17), for small $\epsilon > 0$, we have

$$\frac{v_x}{V_{\epsilon y}^2} > 0 \quad \text{on } (x_\epsilon, y_\epsilon) \in \Sigma_4.$$

This is a contradiction. We complete the proof of Lemma 6. □

Lemma 7. *Let $v = |\nabla V_\epsilon|^2$. If the maximum value of v can be obtained on $\Sigma_1 = \{-1 \leqslant x \leqslant 0$: $y = 0\}$, then there exists $M_7 > 0$, independent on ϵ, such that $|v| < M_7$.*

Proof. From Lemma 2 and [19, Theorem 1], we complete the proof of Lemma 7. □

Lemma 8. *Let $v = |\nabla V_\epsilon|^2$. If the maximum value of v can be obtained on $\Sigma_3 = \{(x,y)$: $-1 < x < 0$, $y = b\sqrt{1-x^2/a^2}\}$, then there exists $M_8 > 0$, independent on ϵ, such that $|v| < M_8$.*

First, we consider following problem:

$$\epsilon r^2 V_{\epsilon rr} + \left(1-r^2\right)V_{\epsilon rr} + \frac{1}{r^2}V_{\epsilon\alpha\alpha} + \frac{1}{r}V_{\epsilon r} - \epsilon V_\epsilon = 0, \quad (r,\alpha) \in D, \tag{4.18}$$

$$N_1 V_\epsilon = 0 \quad \text{on } \alpha = \frac{\pi}{2}, \tag{4.19}$$

$$V_\epsilon = V_\epsilon|_{r=1/d} \quad \text{on } r = \frac{1}{d}, \tag{4.20}$$

$$V_\epsilon = V^{(\theta)}(\alpha) \equiv V_1 - V_0 \quad \text{on } r = 1, \tag{4.21}$$

$$N_2 V_\epsilon = 0 \quad \text{on } r = -\frac{1}{a\cos\alpha}, \tag{4.22}$$

where $\pi/2 < \alpha_0 < \pi$, $\tan\alpha_0 = -\sqrt{a^2-1}$, (r_d, α_d) is the intersection point of $r = 1/d$ $(d > 1)$ and $ar\cos\alpha = -1$;

$$D = \left\{(r,\alpha)\colon\ \frac{1}{d} < r < 1,\ \frac{\pi}{2} < \alpha \leqslant \alpha_0\right\} \cup \left\{(r,\alpha)\colon\ \frac{1}{d} < r < -\frac{1}{a\cos\alpha},\ \alpha_0 \leqslant \alpha < \alpha_d\right\};$$

and

$$N_1 = \left(-\frac{\cos\alpha}{a} + \epsilon\frac{\sin\alpha}{b}\right)\partial_r + \left(\frac{\sin\alpha}{ar} + \epsilon\frac{\cos\alpha}{br}\right)\partial_\alpha - \epsilon,$$

$$N_2 = \left(q\frac{\cos\alpha}{a} + sr\sin^2\alpha\right)\partial_r - \left(q\frac{\sin\alpha}{ar} - s\sin\alpha\cos\alpha\right)\partial_\alpha - (s - q\epsilon).$$

From the classical theory of elliptic equations, problem (4.18)–(4.22) has a unique classical solution.

We choose $u = M(1 - r + \delta)e^{\mu\sqrt{\alpha-\pi/2+\delta^3}} + V^{(\theta)}(1,\alpha)$ $(\mu < 0, \delta > 0)$, then

$$u_r = -Me^{\mu\sqrt{\alpha-\pi/2+\delta^3}}, \qquad u_\alpha = \frac{M(1-r+\delta)\mu}{2\sqrt{\alpha-\pi/2+\delta^3}}e^{\mu\sqrt{\alpha-\pi/2+\delta^3}} + V^{(\theta)}_\alpha,$$

$$u_{rr} = 0, \qquad u_{r\alpha} = -\frac{M\mu}{2\sqrt{\alpha-\pi/2+\delta^3}}e^{\mu\sqrt{\alpha-\pi/2+\delta^3}},$$

$$u_{\alpha\alpha} = V^{(\theta)}_{\alpha\alpha} + \frac{M(1-r+\delta)\mu^2}{4(\alpha-\pi/2+\delta^3)}e^{\mu\sqrt{\alpha-\pi/2+\delta^3}} - \frac{(1-r+\delta)\mu}{4(\alpha-\pi/2+\delta^3)^{3/2}}e^{\mu\sqrt{\alpha-\pi/2+\delta^3}}.$$

If θ is smaller, we can choose η_0 bigger, b smaller and $d > 1$ near to 1. It is easy to obtain that, if M is large enough and independent on δ, $\mu = -\delta^5$, and δ is small enough, then

$$\epsilon r^2 u_{rr} + (1 - r^2)u_{rr} + \frac{1}{r^2}u_{\alpha\alpha} + \frac{1}{r}u_r - \epsilon u \leqslant 0, \quad (r,\alpha) \in D, \tag{4.23}$$

$$N_1 u \leqslant 0 \quad \text{on } \alpha = \frac{\pi}{2}, \tag{4.24}$$

$$u \geqslant V_\epsilon|_{r=1/d} \quad \text{on } r = \frac{1}{d}, \tag{4.25}$$

$$u \geqslant V^{(\theta)}(\alpha) \quad \text{on } r = 1, \tag{4.26}$$

$$N_2 u \leqslant 0 \quad \text{on } r = -\frac{1}{a\cos\alpha}, \tag{4.27}$$

where we have used $(a^2 - 1)s > q$.

Then, from the comparison principle, we have

$$V_\epsilon - V^{(\theta)}(\alpha) \leqslant M(1-r)e^{\sqrt{\alpha-\pi/2}/M^2} \quad \text{and} \quad \left|V_\epsilon - V^{(\theta)}(\alpha)\right| \leqslant M(1-r)e^{\sqrt{\alpha-\pi/2}/M^2}.$$

i.e.,

$$|V_{\epsilon r}| \leqslant CM. \tag{4.28}$$

Proof of Lemma 8. First, we notice that the translation

$$\begin{cases} x = ar\cos\alpha, \\ y = br\sin\alpha \end{cases}$$

can translate (4.1)–(4.5) to (4.18)–(4.22) on the $\Omega'' = \{(x, y)\colon -1 < x < 0,\ b\sqrt{1/d^2 - x^2/a^2} < y < b\sqrt{1 - x^2/a^2}\} \cap \Omega$; then from (4.28), we complete the proof of Lemma 8. □

Corollary. *For any $\epsilon > 0$, there exists $M > 0$, independent on ϵ, such that*

$$|\nabla V_\epsilon| \leqslant M.$$

5. Main theorem and its proof

Main theorem. The problem (3.2) has a unique classical solution $V \in C^2(\Omega) \cap C^{0+1}(\bar{\Omega})$ and V is Lipschitz continuous up to boundary of Ω.

Proof. From Lemmas 1–8 and the Corollary, we only prove the uniqueness.

If there exist two solutions V_1 and V_2, then $w = V_1 - V_2$ satisfies

$$a_{11}\partial_{11}^2 w + 2a_{12}\partial_{12}^2 w + a_{22}\partial_{22}^2 w = 0 \quad \text{in } \Omega, \tag{5.1}$$

$$\partial_2 w = 0 \quad \text{on } y = 0, \tag{5.2}$$

$$\partial_1 w = 0 \quad \text{on } x = 0, \tag{5.3}$$

$$w = 0 \quad \text{on } \bar{\Sigma}_3, \tag{5.4}$$

$$q\partial_1 w + sy\partial_2 w - sw = 0 \quad \text{on } x = -1. \tag{5.5}$$

Multiplying w on two sides of (5.1) and integrating by parts in the Ω, we get

$$\iint_\Omega \big(a_{11}w_x^2 + 2a_{12}w_x w_y + a_{22}w_y^2\big)\,dx\,dy + \iint_\Omega \big((a_{11})_x w w_x + (a_{12})_y w w_x\big)\,dx\,dy$$
$$+ \iint_\Omega \big((a_{12})_x w w_y + (a_{22})_y w w_y\big)\,dx\,dy - \iint_\Omega b_1 w w_x\,dx\,dy$$
$$- \int_{\partial\Omega} (a_{11} w w_x n_x + a_{12} w w_x n_y)\,dl - \int_{\partial\Omega} (a_{12} w w_y n_x + a_{22} w w_y n_y)\,dl = \sum_{i=1}^5 I_i = 0,$$

where (n_x, n_y) is the outer normal vector of $\partial\Omega$.

$$I_1 = \iint_\Omega \big(a_{11}w_x^2 + 2a_{12}w_x w_y + a_{22}w_y^2\big)\,dx\,dy \geqslant 0,$$

$$I_2 = \iint_\Omega \big((a_{11})_x w w_x + (a_{12})_y w w_x\big)\,dx\,dy = -3\iint_\Omega x w w_x\,dx\,dy$$
$$= \frac{3}{2}\iint_\Omega w^2\,dx\,dy - \frac{3}{2}\int_{\partial\Omega} x w^2 n_x\,dl = \frac{3}{2}\iint_\Omega w^2\,dx\,dy - \frac{3}{2}\int_{\Sigma_4} w^2\,dy,$$

$$I_3 = \iint\limits_{\Omega} \big((a_{12})_x w w_y + (a_{22})_y w w_y\big)\, dx\, dy = -3 \iint\limits_{\Omega} y w w_y\, dx\, dy$$
$$= \frac{3}{2} \int\limits_{\Omega} w^2\, dx\, dy - \frac{3}{2} \int\limits_{\partial\Omega} y w^2 n_y\, dl = \frac{3}{2} \iint\limits_{\Omega} w^2\, dx\, dy,$$
$$I_4 = -\int\limits_{\partial\Omega} (a_{11} w w_x n_x + a_{12} w w_x n_y)\, dl$$
$$= \int\limits_{\Sigma_1} a_{12} w w_x\, dx - \int\limits_{\Sigma_2} a_{11} w w_x\, dy + \int\limits_{\Sigma_4} a_{11} w w_x\, dy,$$

by boundary conditions (5.3) and (5.4), we have

$$I_4 = \int\limits_{\Sigma_1} a_{12} w w_x\, dx - \int\limits_{\Sigma_4} \frac{a_{11} s y}{q} w w_y\, dy + \int\limits_{\Sigma_4} \frac{a_{11} s}{q} w^2\, dy$$
$$= \int\limits_{\Sigma_4} \frac{c_0^2 s}{\rho_0 u_\infty V_0} w^2\, dy + \frac{1}{2} \int\limits_{\Sigma_4} \frac{c_0^2 s}{\rho_0 u_\infty V_0} w^2\, dy,$$
$$I_5 = -\int\limits_{\partial\Omega} (a_{12} w w_y n_x + a_{22} w w_y n_y)\, dl$$
$$= \int\limits_{\Sigma_1} a_{22} w w_y\, dx - \int\limits_{\Sigma_2} a_{12} w w_y\, dy + \int\limits_{\Sigma_4} a_{12} w w_y\, dy,$$

by boundary condition (5.2), we have

$$I_5 = \int\limits_{\Sigma_4} y w w_y\, dy = -\frac{1}{2} \int\limits_{\Sigma_4} w^2\, dy,$$

noticing

$$\frac{c_0^2 s}{\rho_0 u_\infty V_0} > 1, \qquad \frac{V_0^2}{c_0^2} \frac{\rho_0}{\rho_\infty} < 1 + \frac{V_0^2}{c_0^2} < 2, \qquad \frac{c_0^2 s}{\rho_0 u_\infty V_0} > \frac{3}{2},$$

we get: if $w \not\equiv 0$, then

$$0 = \sum_{i=1}^{6} I_i \geqslant \sum_{i=2}^{6} I_i \geqslant \frac{3}{2} \iint\limits_{\Omega} w^2\, dx\, dy > 0.$$

Hence $w \equiv 0$ and $V_1 = V_2$. We complete the proof of the main theorem. □

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Appendix A

Suppose the equation of shock before meeting the ramp is $x = \sigma t$, here $\sigma > 0$ is a constant. Namely, the shock moves forward by the speed σ. Under the self-similar coordinates $\xi = x/t$ and $\eta = y/t$, the point of reflection of the oblique shock is $(\sigma, \sigma/\operatorname{tg}\theta)$. Denoting by $k(\theta) > 0$ the slope of the reflected oblique shock, its equation is represented as

$$\xi - \sigma = k(\theta)\left(\eta - \frac{\sigma}{\operatorname{tg}\theta}\right).$$

If we write the velocity and density behind the reflected shock as $(u(\theta), v(\theta))$ and $\rho(\theta)$, respectively, then by the Bernoulli law in (2.1) we have

$$\phi(\xi,\eta) = u(\theta)\xi + v(\theta)\eta - h\big(\rho(\theta)\big) - \frac{1}{2}\big(u^2(\theta) + v^2(\theta)\big).$$

In terms of the continuity condition of potential ϕ on the reflected shock, we have

$$u(\theta)k(\theta) + v(\theta) = u_\infty k(\theta), \tag{A.1}$$

$$\sigma\left(1 - \frac{k(\theta)}{\operatorname{tg}\theta}\right)\big(u_\infty - u(\theta)\big) + h\big(\rho(\theta)\big) + \frac{1}{2}\big(u^2(\theta) + v^2(\theta)\big) = h(\rho_\infty) + \frac{u_\infty^2}{2}. \tag{A.2}$$

In addition, it follows from the Rankine–Hugoniot condition on the reflected shock that

$$\rho(\theta)u(\theta) - \rho_\infty u_\infty - \sigma\left(1 - \frac{k(\theta)}{\operatorname{tg}\theta}\right)\big(\rho(\theta) - \rho_\infty\big) - k(\theta)\rho(\theta)v(\theta) = 0. \tag{A.3}$$

Since the velocity of the flow is tangent to the wall, then one derives

$$u(\theta) = v(\theta)\operatorname{tg}\theta. \tag{A.4}$$

Finally, the physical entropy condition is satisfied,

$$\rho_\infty < \rho(\theta). \tag{A.5}$$

Now we start to do some analysis on $u(\theta)$, $v(\theta)$ and $\rho(\theta)$ in terms of (A.1)–(A.5). Set

$$\begin{cases} u(\theta) = u_0\theta^2 + u_\theta\theta^3, \\ v(\theta) = v_0\theta + v_\theta\theta^2, \\ \rho(\theta) = \rho_0 + \rho_\theta\theta, \\ k(\theta) = k\theta + k_\theta\theta^2, \end{cases} \tag{A.6}$$

where $u_0 = 0$, $v_0 = 0$, $\rho_0 > 0$ (normal shock reflect parament [6]) and $k > 0$ are the determined constants, u_θ, v_θ, ρ_θ and k_θ are the determined functions of θ.

Substituting (A.6) into (A.1)–(A.5) and comparing the coefficients of θ on two sides yields

$$\begin{cases} \sigma(1-k)(\rho_0 - \rho_\infty) + \rho_\infty u_\infty = 0, \\ h(\rho_0) = h(\rho_\infty) - \sigma(1-k)u_\infty + \frac{u_\infty^2}{2}, \\ \rho_\infty < \rho_0. \end{cases} \tag{A.7}$$

Eliminating ρ_0 in (A.7), we get an equation on k:

$$F(k) = 0,$$

where

$$F(k)=\sigma(1-k)\left\{\left(\rho_\infty^{\gamma-1}+\frac{\gamma-1}{2A\gamma}u_\infty^2-\sigma(1-k)u_\infty\right)^{\frac{1}{\gamma-1}}-\rho_\infty\right\}+\rho_\infty u_\infty.$$

Next we show that $F(k)=0$ has a unique solution in $(1,\infty)$. Indeed, it is easy to verify that

$$F(1)=\rho_\infty u_\infty>0,\qquad \lim_{k\to\infty}F(k)=-\infty\quad\text{and}\quad F'(k)>0\quad\text{for } k\in(1,\infty).$$

Denote by $V_0=\sigma(k-1)$ and $c_0=c(\rho_0)$, here $k>1$ is the solution of $F(k)=0$. Then we have:

Lemma A.1. *If to denote the Mach number of coming flow by $M_\infty=u_\infty/c_\infty$, for $1<\gamma\leqslant 2$ and $1<M_\infty\leqslant 6/5$, then V_0 and c_0 satisfy the following relations*:

$$V_0u_\infty=h(\rho_0)-h(\rho_\infty)-\frac{u_\infty^2}{2},\qquad V_0(\rho_0-\rho_\infty)=\rho_\infty u_\infty,\tag{A.8}$$

and

$$\frac{V_0^2}{c_0^2}<1,\qquad \frac{V_0^2}{c_0^2}\left(\frac{\rho_0}{\rho_\infty}-1\right)<1.\tag{A.9}$$

Remark A.1. The conclusion (A.9) is similar to the uniform stability condition for the weak shock in [22,23].

Proof of Lemma A.1. Obviously, (A.8) comes from (A.7) directly.

Firstly, we show

$$\frac{V_0^2}{c_0^2}\left(\frac{\rho_0}{\rho_\infty}-1\right)<1.$$

It follows from (A.8) that

$$\frac{V_0^2}{c_0^2}\left(\frac{\rho_0}{\rho_\infty}-1\right)=\frac{V_0u_\infty}{c_0^2},$$

one only needs to prove $V_0u_\infty/c_0^2<1$. We intend to use the contradictory method to show this.

Otherwise, suppose $V_0u_\infty\geqslant c_0^2$. By the first formula in (A.8) we have

$$\frac{c_0^2}{c_\infty^2}\geqslant\frac{1+\frac{\gamma-1}{2}M_\infty^2}{2-\gamma}.\tag{A.10}$$

In light of the second formula in (A.8) and the assumption $V_0u_\infty\geqslant c_0^2$, one has

$$\frac{c_0^2}{c_\infty^2}\left(\frac{\rho_0}{\rho_\infty}-1\right)\leqslant M_\infty^2.\tag{A.11}$$

Combining (A.10) with (A.11) and using $c_0^2/c_\infty^2=(\rho_0/\rho_\infty)^{\gamma-1}$ yields

$$\frac{1+\frac{\gamma-1}{2}M_\infty^2}{2-\gamma}\leqslant\left(1+\frac{c_\infty^2M_\infty^2}{c_0^2}\right)^{\gamma-1}\leqslant\left(1+\frac{(2-\gamma)M_\infty^2}{1+\frac{\gamma-1}{2}M_\infty^2}\right).\tag{A.12}$$

But (A.12) does not hold for $1 < \gamma < 2$ and $1 < M_\infty \leqslant 1.2$. Indeed, if we set $s = \gamma - 1$, then in terms of (A.12) we define the function

$$f(s) = \frac{2 + M_\infty^2 s}{2(1-s)} - g(s), \quad \text{where } g(s) = \left(1 + \frac{2M_\infty^2(1-s)}{2 + M_\infty^2 s}\right)^s \quad \text{for } s \in (0,1].$$

A direct computation yields $f(0) = 0$, and

$$\begin{aligned} f'(s) &= \frac{2 + M_\infty^2}{2(1-s)^2} \\ &\quad + g(s)\left\{\frac{2(2M_\infty^2 + M_\infty^4)s}{(2 + M_\infty^2 s)^2 + 2M_\infty(1-s)(2 + M_\infty^2 s)} - \ln\left(1 + \frac{2(1-s)M_\infty^2}{2 + M_\infty^2 s}\right)\right\} \\ &\geqslant \frac{2 + M_\infty^2}{2(1-s)^2} + g(s)\left(\frac{2s}{3} - (1-s)M_\infty^2\right). \end{aligned}$$

Obviously, when $s \geqslant 3M_\infty^2/(2 + 3M_\infty^2)$, one derives

$$f'(s) > 0.$$

When $s < 3M_\infty^2/(2 + 3M_\infty^2)$ and $1 < M_\infty \leqslant 1.2$, we still have

$$f'(s) \geqslant 1 + \frac{M_\infty^2}{2} - (1 + M_\infty)^{3M_\infty^2/(2+3M_\infty^2)} \ln\left(1 + M_\infty^2\right) > 0.$$

Thus (A.12) does not hold and $(V_0^2/c_0^2)(\rho_0/\rho_\infty - 1) < 1$ is proved.

Next, we show $V_0^2/c_0^2 < 1$. Since $u_\infty V_0 < c_0^2$, then it follows from the second formula in (A.8) that

$$\frac{V_0^2}{c_0^2} < \frac{V_0^2}{u_\infty^2}\left(\frac{\rho_0}{\rho_\infty} - 1\right). \tag{A.13}$$

In terms of the physical entropy condition we have $u_\infty > c_0$, hence

$$\frac{V_0^2}{c_0^2} < \frac{V_0^2}{c_0^2}\left(\frac{\rho_0}{\rho_\infty} - 1\right) < 1.$$

Then we complete the proof of Lemma A.1. $\square$

Using Lemma A.1, direct computing, we have:

Lemma A.2. *Let*

$$s = \rho_0 - \rho_\infty + \frac{\rho_0 u_\infty V_0}{c_0^2}, \qquad q = \frac{\rho_0 u_\infty}{V_0}\left(1 - \frac{V_0^2}{c_0^2}\right) > 0, \qquad a = \frac{c_0^2}{V_0^2}.$$

Then

$$\left(a^2 - 1\right)s > q. \tag{A.14}$$

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